$L = D - A \qquad D = \begin{pmatrix} d(v_1) & & \\ & \ddots & \\ & & d(v_n) \end{pmatrix}$

$A =$ adjacency matrix

1) Show $\vec{1} = [1, \dots, 1]^T$ is an eigenvector:

$$(L\vec{1})_i = d(v_i) - \sum_{j=1}^{n} A_{ij} = 0 \quad \text{since } A_{ij} = \#\{\text{edges between } i,j\}$$

$$\Rightarrow \quad L\vec{1} = 0 \cdot \vec{1}$$

$$\det(L) = \prod_{i=1}^{n} \lambda_i \quad \text{where } \lambda_i \text{ are the } n\text{-eigenvalues of } L$$

$$= 0 \quad \text{since } 0 \text{ is always an eval.}$$

2) If G is d regular $D - A = dI - A$

The characteristic polynomial of A is

$$p_A(z) = \det(zI - A) = \prod_{i=1}^{k} (z - \lambda_i)^{m(i)}$$ where $m(i)$ is the multiplicity of λ_i

$$\sum_{i=1}^{k} m(i) = n$$

This is by the fundamental thm of algebra that says we have n roots for a degree n polynomial.

Then $p_L(z) = \det\left((z-d)I + A\right) = (-1)^n \prod_{i=1}^{k} \det\left[(d-z)I - A\right]$

$$= (-1)^n \prod_{i=1}^{k} \left((d-z) - \lambda_i\right)^{m(i)}$$

By the fundamental theorem of algebra, we're done since this accounts for all n roots of p_L

It remains to prove that the dimensions of the eigenspaces for $d-\lambda_i$ and λ_i are the same. This is easy: v is an eigenvector of L with eigenvalue $d-\lambda_i$ iff v is an eigenvector of A.

$$Lv = (dI - A)v = (d-\lambda_i)v$$

$$\Leftrightarrow \cancel{dv} - Av = -\lambda_i v + \cancel{dv}$$

$\Leftrightarrow$ v is an eigenvector of A for eigenvalue λ_i

This shows that the "geometric multiplicity" of λ_i and $d-\lambda_i$ are the same.